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A Note on Symmetry Boundary Conditions in Finite Element Methods

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Abstract. A short discussion on two kinds of symmetry boundary conditions in the context of variational formulations and finite element methods is presented. Applications including modeling of open boundaries in computational fluid mechanics are discussed and a numerical example is presented.

1. Symmetry Boundary Conditions of the First and Second Kind

We begin with a model elliptic system related, for example, to the deformation of a two-dimensional domain depicted in Figure 1. The motion in the top figure amounts to what we shall call the physical symmetry. As a result of a symmetry of load and kinematic boundary conditions, the motion becomes symmetric with respect to the y axis. In terms of the solution vector $\boldsymbol{u} = (u, v)$, we have

$$u(x,y) = -u(-x,y); \qquad v(x,y) = v(-x,y). \tag{1.1}$$

The second condition implies

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial v}{\partial x}(-x,y). \tag{1.2}$$

Passing with $x \to 0$ in the first of conditions (1.1) and in (1.2), we get the *symmetry boundary conditions of the first kind*

$$u = 0; \quad \frac{\partial v}{\partial x} = 0, \qquad \text{for } x = 0, \tag{1.3}$$

or, equivalently,

$$u = 0; \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \qquad \text{for } x = 0. \tag{1.4}$$

The problem can be solved in half of the domain Ω with the symmetry boundary conditions (1.3) or (1.4) along the y axis. The situation is different in the second example in Figure 1. As a result of the enforced uniform motion along the top and bottom lines, another kind of symmetry can be observed. In terms of the components, we have

$$u(x,y) = u(-x,y); \qquad v(x,y) = v(-x,y). \tag{1.5}$$

This leads again to a boundary condition on the derivative of the solution

$$\frac{\partial u}{\partial x} = 0, \qquad \left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0\right), \quad \text{for } x = 0. \tag{1.6}$$

We call (1.6) the *symmetry boundary conditions of second kind.*

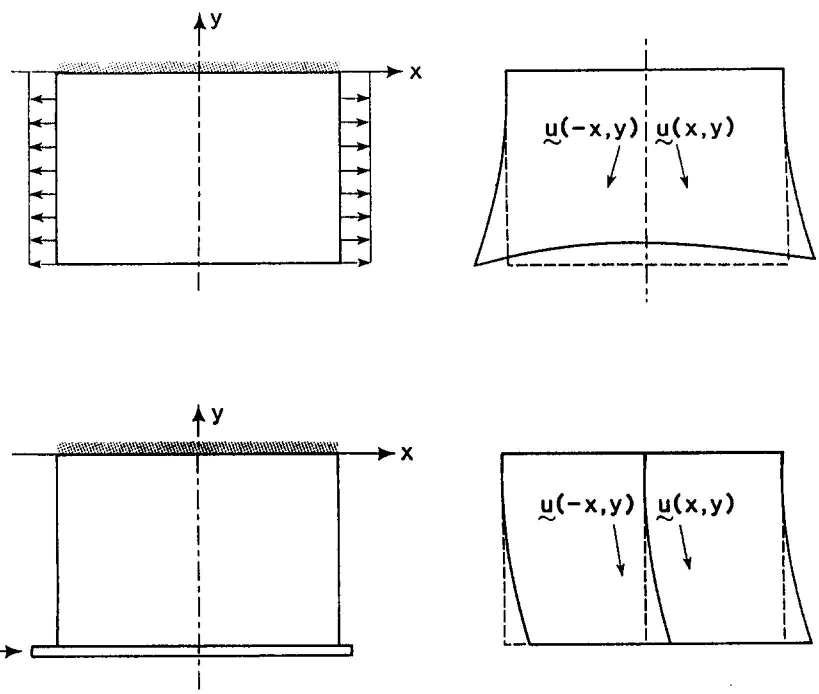

Figure 1. Symmetry boundary conditions of the first and second kind.

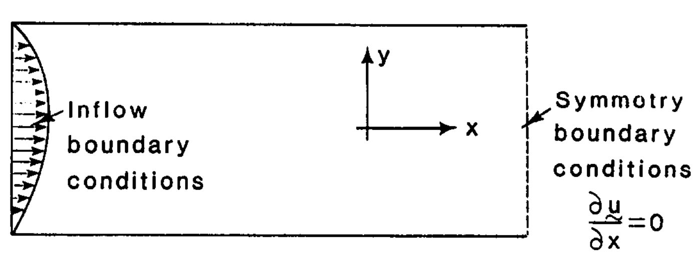

Figure 2.

Formally the elliptic system can be again solved in the half of the domain Ω, this time with boundary conditions (1.6) along the $x = 0$ axis.

The interesting difference between the two types of symmetry boundary conditions appears when constructing the variational formulation of the problems. Multiplying the equations of equilibrium by a test function $\delta \boldsymbol{u}$ and integrating by parts, we arrive at the usual boundary term of the form

$$\boldsymbol{t} \cdot \delta \boldsymbol{u} = t_x\, \delta u + t_y\, \delta v, \tag{1.7}$$

where $\boldsymbol{t}$ is the stress vector. Decomposing both the stress vector and the virtual displacement $\delta \boldsymbol{u}$ into normal and tangential components (to the boundary), we get

$$\boldsymbol{t} \cdot \delta \boldsymbol{u} = \sigma_n\, \delta u_n + \sigma_\tau\, \delta u_\tau, \tag{1.8}$$

where σ_n and σ_τ are the normal and tangential stresses and δu_n and δu_τ are the normal and tangential virtual displacements. Eliminating the normal virtual displacement

$$\delta u_n = 0, \tag{1.9}$$

we arrive at the boundary term

$$\sigma_\tau \, \delta u_\tau \tag{1.10}$$

which disappears according to the second of conditions (1.4). In particular, the second of the boundary conditions (1.4) is identified as the *natural boundary condition* corresponding to the governing linear elliptic operator.

The situation is quite different in the second case. Rewriting the boundary term explicitly in terms of derivatives of $\boldsymbol{u}$, we get

$$\sigma_{xx}\,\delta u + \sigma_{xy}\,\delta v = (2\mu+\lambda)\frac{\partial u}{\partial x}\,\delta u + \lambda\frac{\partial v}{\partial y}\,\delta u + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\delta v. \tag{1.11}$$

Accounting now for boundary conditions (1.6), we reduce (1.11) to

$$\lambda\frac{\partial v}{\partial y}\,\delta u + \mu\frac{\partial u}{\partial y}\,\delta v. \tag{1.12}$$

Note the fact that (1.12) still *does* include the contributions of the solution vector (u,v) and it will contribute to the stiffness matrix. These remaining boundary terms indicate that the symmetry boundary conditions of second kind cannot be classified as *natural* boundary conditions.

REMARK. The boundary terms (1.12) are perfectly legitimate from the mathematical point of view. It follows from the identity

$$\int_\Omega\left(-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)dx\,dy = \int_{\partial\Omega}\left(\frac{\partial u}{\partial x}(-n_y) + \frac{\partial u}{\partial y}n_x\right)v = \frac{\partial u}{\partial\tau}v, \tag{1.13}$$

that the tangential derivative $\partial u/\partial\tau$ is a well defined element of $H^{-\frac{1}{2}}(\partial\Omega)$ for all functions u from $H^1(\Omega)$ (comp. the generalized Green's formula in [1]).

What remains to be verified is the coerciveness of the resulting bilinear form. In the example shown, for instance, substituting $\delta\boldsymbol{u} = \boldsymbol{u}$ in (1.12) we get the boundary term of the form

$$\int_{\partial\Omega}\left(\lambda\,u\frac{\partial v}{\partial y} + G\frac{\partial u}{\partial y}v\right)ds. \tag{1.14}$$

In the special case $\lambda = G$ this term reduces to

$$\lambda\int_{\partial\Omega}\frac{\partial}{\partial y}(uv)\,ds = 0, \tag{1.15}$$

due to the kinematic boundary conditions imposed on the top and bottom of the plate.

2. Modification of the Bilinear Form Without Changing the Differential Operator

Including additional boundary integrals of the type (1.12) in the definition of the bilinear form may be equivalent to adding extra terms in the domain integral contribution to the bilinear form in such a way that the corresponding differential operator remains unchanged. An example of such a procedure is given in [1]. Defining the bilinear form as

$$B(u,v) = \int_\Omega\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + c\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - c\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dx\,dy, \tag{2.1}$$

with c a constant we easily identify the corresponding operator as the Laplace operator Δu and the boundary operator in the form of the *oblique derivative*

$$\frac{\partial u}{\partial n} + c\,\frac{\partial u}{\partial\tau}. \tag{2.2}$$

The differential operator is in particular *independent of* c, and the form is easily proved to be coercive under regular assumptions.

3. AN APPLICATION IN FLUID MECHANICS

Symmetry boundary conditions of the second kind prove to be very useful in modeling *open boundary conditions* in fluid mechanics. The exterior flow problems are usually formulated in unbounded domains. For practical calculations a finite computational domain must be selected and, as a result, an artificial *open boundary* is introduced. Consequently, relevant boundary conditions must be specified on such boundaries.

An example for an application of the symmetry boundary conditions of second type would be a flow in an infinite horizontal duct. Assuming that, at a far enough distance, flow stabilizes, the symmetry boundary conditions may be implemented on the exit boundary. Notice that *no extra information* is necessary to apply these conditions.

The symmetry boundary conditions have been successfully applied to model much more complex open boundary conditions resulting in perfect "*fully absorbing*" treatment of boundary terms. Figure 3 shows an example of a solution to the steady state Euler equations by means of the Taylor-Galerkin method decribed in [2] (flow over a wedge problem). The top line of the domain is identified as a subsonic inflow/subsonic outflow boundary, gradually changing into supersonic outflow, and the right-hand side is a supersonic outflow boundary. The symmetry boundary conditions applied along both boundaries result in no artificial reflection of the shock from the upper outflow boundary.

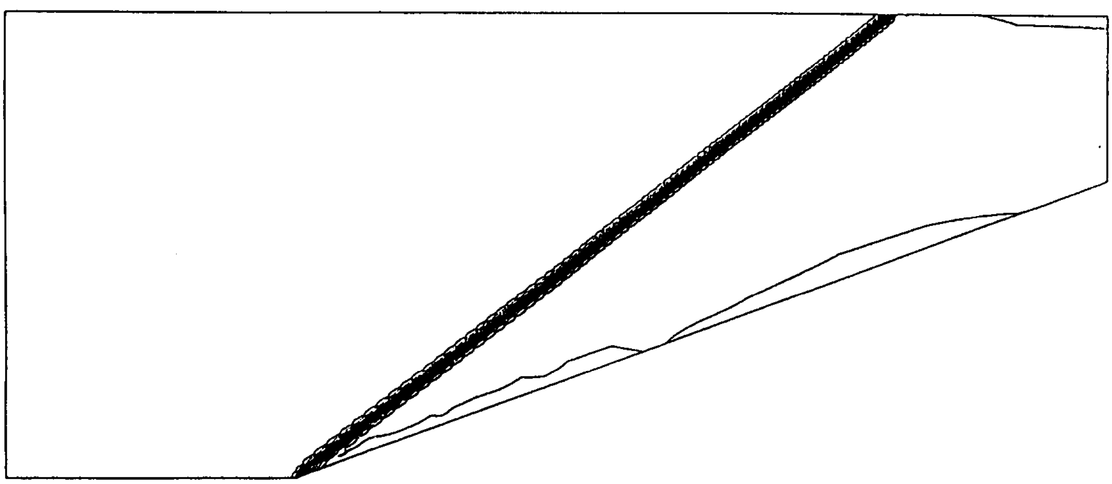

Figure 3. Flow over a wedge problem for Euler equations.

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